

Static Optimization with MATLAB

Day 3: MATLAB Constrained Optimization

GSEFM Pre-Semester Course

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fmincon

`fmincon` is MATLAB's constrained optimization routine. It is a general purpose solver for nonlinear constrained optimization problems.

- look up documentation via `doc fmincon`
- look up source code via `edit fmincon` or `open fmincon`

Coding Exercise 1

Implement and solve the following constrained optimization problem:

$$\begin{aligned} & \max_{(x,y) \in \mathbb{R}^2} (x - y^2) \\ & s.t. \ x^2 + y^2 = 4 \\ & \quad x \geq 0, \ y \geq 0 \end{aligned}$$

You can check your solution by looking at your theoretical solution (see Slides about Optimization with Mixed Constraints).

Coding Exercise 2

Implement and solve the following constrained optimization problem:

$$\begin{aligned}\Pi(x_1, x_2) &= \left(100 - \frac{3}{2}x_1\right)x_1 + (50 - 2x_2)x_2 - 10(x_1 + x_2) \\ \text{s.t. } x_1 + x_2 &\leq 33 \quad x_1 \leq 27, \quad x_2 \leq 27\end{aligned}$$

You can check your solution by looking at your theoretical solution (see Slides about Optimization with Mixed Constraints).

Penalty Method I

Idea: Transform the constrained optimization problem into an unconstrained optimization problem by adding a penalty term to the objective function:

$$\min_{x \in \mathbb{R}^n} f(x) + \frac{\alpha}{2} \sum_{i=1}^m h_i(x)^2 + \frac{\alpha}{2} \sum_{i=1}^l (\max\{0, g_i(x)\})^2$$

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Penalty Method II

1. Choose $\alpha^0 > 0$ and $x^0 \in \mathbb{R}^n$; set $k = 0$.
 2. Solve the unconstrained optimization problem and obtain current iterate x^k
 3. If $h(x^k) = 0$, STOP. Otherwise, determine $\alpha^{k+1} > \alpha^k$.
 4. Set $k = k + 1$ and go to step 2.
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- In practice, you can stop if e.g. $\sqrt{\sum_{i=1}^m h_i(x)^2} \leq 10^{-4}$
 - $\alpha^0 = 1$ and $\alpha^{k+1} = 10\alpha^k$ could be a choice
 - Note, even if g and h are twice continuously differentiable, the penalty function is at most only once $\Rightarrow$ hand over gradient and use e.g. BFGS (Quasi-Newton Method).

Lagrange-Newton Method I

Idea: Tackle directly the KKT-system of the constrained optimization problem. To take care of the inequality constraints, the non-negative LM, and the complementary slackness condition, we use a so-called NCP-function.

Definition (NCP function)

A function $\varphi : \mathbb{R}^2 \rightarrow \mathbb{R}$ is called a NCP function if it holds that

$$\varphi(a, b) = 0 \Leftrightarrow a \geq 0, b \geq 0, ab = 0.$$

Lagrange-Newton Method II

Examples for NCP functions are

- Minimum function

$$\varphi(a, b) = 2 \min\{a, b\}$$

- Fischer-Burmeister function

$$\varphi(a, b) = a + b - \sqrt{a^2 + b^2}$$

- Penalized Minimum function

$$\varphi(a, b) = 2\lambda \min\{a, b\} + (1 - \lambda) \max\{a, 0\} \max\{b, 0\}$$

for $\lambda \in (0, 1)$ fixed.

- Penalized Fischer-Burmeister function

$$\varphi(a, b) = \lambda(a + b - \sqrt{a^2 + b^2}) + (1 - \lambda) \max\{a, 0\} \max\{b, 0\}$$

for $\lambda \in (0, 1)$ fixed.

Lagrange-Newton Method I

From this, we obtain the following nonlinear system of equations:

$$0 = \Phi(x, \lambda) = \begin{pmatrix} \nabla f(x) + \nabla h(x)^T \lambda + \nabla g(x)^T \mu \\ h(x) \\ \varphi(-g(x), \lambda) \end{pmatrix}$$

- The Lagrange-Newton Method is a Newton-type method for solving this system of equations.
- Since the NCP functions are however not everywhere differentiable, you have to take care of a lot of other things (not mentioned here)
- Drawback: Since we only solve for KKT system, this might be more suited for convex optimization problems.

SQP Method

Idea: Reformulate your problem at every iteration into a quadratic programming problem (QP) and solve it. Since you solve every iteration sequentially, this is called Sequential Quadratic Programming (SQP). This type of method is probably the most popular method for solving constrained optimization problems.

- The reformulation is (no derivation):

$$\begin{aligned} \max_d \quad & \nabla f(x)^T d + \frac{1}{2} d^T \nabla^2 \mathcal{L}(x) d \\ \text{s.t.} \quad & h(x) + \nabla h(x)^T d = 0, \quad g(x) + \nabla g(x)^T d \leq 0 \end{aligned}$$

- Update then $x^+ = x + d$

Coding Exercise 3

Implement and solve the following constrained optimization problem **using a self-implemented penalty method**:

$$\begin{aligned} \min_{(x_1, x_2, x_3) \in \mathbb{R}^3} \quad & 1000 - x_1^2 - 2x_2^2 - x_3^2 - x_1x_2 - x_1x_3 \\ \text{s.t.} \quad & x_1^2 + x_2^2 + x_3^2 = 25 \\ & 8x_1 + 14x_2 + 7x_3 - 56 = 0. \end{aligned}$$

Hint: You may use `fminunc`, the solution is $(x_1^*, x_2^*, x_3^*) = (3.512, 0.127, 3.552)$, you might use the initial guess $(x_1^0, x_2^0, x_3^0) = (3, 0.2, 3)$.

Coding Exercise 4

Implement and solve the following constrained optimization problem **using a self-implemented Lagrange-Newton Method**:

$$\begin{aligned} \min_{(x_1, x_2, x_3) \in \mathbb{R}^3} \quad & 1000 - x_1^2 - 2x_2^2 - x_3^2 - x_1x_2 - x_1x_3 \\ \text{s.t.} \quad & x_1^2 + x_2^2 + x_3^2 = 25 \\ & 8x_1 + 14x_2 + 7x_3 - 56 = 0. \end{aligned}$$

Hint: You may use `fsolve`, the solution is $(x_1^*, x_2^*, x_3^*) = (3.512, 0.127, 3.552)$ and $(\mu_1^*, \mu_2^*) = (1.223, 0.275)$, you might use the initial guess $(x_1^0, x_2^0, x_3^0) = (3, 0.2, 3)$ and $\mu^0 = (0, 0)$.

References I

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